Mathematics for Engineers II. lectures

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Numerics of Differential Equations

This work was supported by the construction EFOP-3.4.3-16-2016-00021. The project was supported by the European Union, co-financed by the European Social Fund.

Successive approximation

$$x'(t) = f(t, x(t)), \qquad x \in I, \tag{1}$$

$$x(\xi) = \eta \tag{2}$$

The initial value problem (1)-(2) is equivalent to the integral equation:

$$x(t) = \underbrace{\eta + \int_{\xi}^{t} f(s, x(s))\,\mathrm{ds}}_{(Tx)(t)}.$$

Operator T fulfils the assumptions of Banach fixed point theorem if f fulfils the assumptions of Picard-Lindelöf theorem. The unique fixed point of T will be the unique solution of the initial value problem (1)-(2). Like in the case of nonlinear equations the nth iterate will be an approximate solution of the initial value problem.

$$x_0(t) = \eta, \quad x_{n+1}(t) = T(x_n(t))$$

Successive approximation

Example

Let's consider the initial value problem

$$x' = x, \quad x(0) = 1.$$

Its unique solution is $x(t) = e^t$. Then $\xi = 0,\ \eta = 1,\ x_0(t) = 1.$

$$x_1(t) = 1 + \int_0^t 1 \,\mathrm{ds} = 1 + t, \quad x_2(t) = 1 + \int_0^t 1 + s \,\mathrm{ds} = 1 + t + \frac{t^2}{2},$$

and

$$x_n(t) = 1 + \int_0^t 1 + s + \cdots + \frac{s^{n-1}}{(n-1)!} \,\mathrm{ds} = 1 + t + \cdots + \frac{t^n}{n!},$$

which is exactly the nth partial sum of e^t.

Successive approximation

Exercises

Find the approximate solutions of the initial value problems using successive approximation!

- $\dot{x} = 2x - t, \quad x(0) = 1,$
- $y' = y(3 - y), \quad x(0) = 1,$
- $y'(t) = -y(t) + \cos(t), \quad y(0) = 0,$
- $\dot{x} = 2x - t, \quad x(1) = 1.$

Reminder

Taylor' theorem: Let $f : [a, b] \to \mathbb{R}$ be an $(n+1)$ times continuously differentiable function on $]a, b[$, the nth derivative is continuous on $[a, b]$, then for arbitrary $\bar{x},\ x \in [a, b]$ there is a β between x and $\bar{x}$ such that

$$f(x) = \underbrace{f(\bar{x}) + \frac{f'(\bar{x})}{1!}(x-\bar{x}) + \cdots + \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n}_{n.\ \text{Taylor polynomial}} + \underbrace{\frac{f^{(n+1)}(\beta)}{(n+1)!}(x-\bar{x})^{n+1}}_{\text{error term}}$$

If in the initial value problem the function f is n times differentiable in a neighbourhood of (ξ, η), then the unknown function x is $(n+1)$ times differentiable in a neighbourhood of ξ and Taylor's theorem entails the existence of β between t and ξ such that

$$x(t) = x(\xi) + \frac{x'(\xi)}{1!}(t-\xi) + \cdots + \frac{x^{(n)}(\xi)}{n!}(t-\xi)^n + \frac{x^{(n+1)}(\beta)}{(n+1)!}(t-\xi)^{n+1}.$$

Taylor series method, Euler's method

Taking into account the previous expression for x, the partial sum

$$\tilde{x}(t) = x(\xi) + \frac{x'(\xi)}{1!}(t-\xi) + \cdots + \frac{x^{(n)}(\xi)}{n!}(t-\xi)^n$$

is an approximate solution of the initial value problem for which

$$|x(t) - \tilde{x}(t)| = \frac{1}{(n+1)!}\left|x^{(n+1)}(\beta)(t-\xi)^{n+1}\right| \leq K\left|(t-\xi)^{n+1}\right| =: \mathcal{O}\left((t-\xi)^{n+1}\right)$$

if the $(n+1)$th derivative is bounded.

Definition

Let $g\colon I \to \mathbb{R}$ and $f\colon I \to \mathbb{R}$ be two functions. One can write $g = \mathcal{O}(f)$ if and only if there is a positive constant K such that for all sufficiently large t

$$|g(t)| \leq K|f(t)|.$$

Taylor series method, Euler's method

Differentiating the ODE one can get different order approximation of the solution of the initial value problem. The firs order approximation is called **Euler's method**.

For example the second order approximation is

$$x(\xi) = \eta, \quad x'(\xi) = f(\xi, \eta), \quad x''(t) = \frac{\partial f(t,x)}{\partial t} + \frac{\partial f(t,x)}{\partial x} f(t,x) \Rightarrow$$

$$x''(\xi) = \frac{\partial f(\xi,\eta)}{\partial t} + \frac{\partial f(\xi,\eta)}{\partial x} f(\xi,\eta).$$

One can derive higher order Taylor approximations in a similar way resulting more complex formulae.

Taylor series method, Euler's method

If $n = 1$, then

$$x(t) = x(\xi) + x'(\xi)\underbrace{(t-\xi)}_{=:h} + \mathcal{O}((t-\xi)^2) = \underbrace{\eta + hf(\xi,\eta)}_{\text{Euler approximation}} + \mathcal{O}(h^2).$$

Euler's method Let's try to approximate the solution of the initial value problem on the interval$[\xi, \bar{t}\ ]$. Let h be given as $h := \frac{\bar{t}-\xi}{n}$ for a fixed $n \in \mathbb{N}$. Than the iteration scheme is the following:

$$\begin{aligned} t_0 &= \xi, & x_0 &= \eta, & \\ t_{i+1} &= t_i + h, & x_{i+1} &= x_i + hf(t_i, x_i), & i &= 0, \dots, n-1. \end{aligned}$$

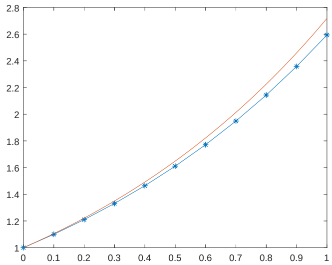
Exercise

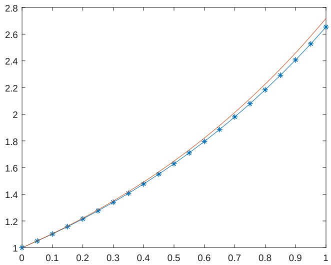
Determine the approximate solution of the following initial value problems using Euler's method on the interval I with step size h!

- $\dot{x}(t) = tx(t), \quad x(0) = 1,\ I = [0, 1],\ h = 0.25,$
- $x' = 3x(1 - x), \quad x(0) = 0.1,\ I = [0, 2.5],\ h = 0.5.$

Example

Approximate solution of the initial value problem $x' = x, \ x(0) = 1$ with Euler's method on the interval $[0, 1]$ with step sizes $h = 0.1$ and $h = 0.05$:





Taylor series method, Euler's method

If $n = 2$, then with the previously defined h we get

$$x(t) = \underbrace{\eta + hf(\xi,\eta) + \frac{h^2}{2!}\left(\frac{\partial f(\xi,\eta)}{\partial t} + f(\xi,\eta)\frac{\partial f(\xi,\eta)}{\partial x}\right)}_{\text{second order Taylor approximation}} + \mathcal{O}((t-\xi)^3)$$

Taylor's method if $n = 2$

Assume that we would like to approximate the solution of an initial value problem on the interval $[\xi, \bar{t}\,]$. Let $h := \frac{\bar{t}-\xi}{n}$ for a fixed $n \in \mathbb{N}$. Then the iteration is the following:

$$t_0 = \xi, \qquad x_0 = \eta,$$

$$t_{i+1} = t_i + h, \qquad x_{i+1} = x_i + hf(t_i,x_i) + \frac{h^2}{2!}\left(\frac{\partial f(t_i,x_i)}{\partial t} + f(t_i,x_i)\frac{\partial f(t_i,x_i)}{\partial x}\right),$$

$i = 0,\dots,n-1.$

Definition

The quantity

$$g_i := \underbrace{\frac{x(t_{i+1}) - x(t_i)}{h}}_{\approx x'(t_i)} - f(t_i, x(t_i))$$

is said to be the **local error of Euler's method**. A numerical method for the initial value problem (1)-(2) is called p**th order consistent in the class** F ($p > 0$), if

$$|g_i| = \mathcal{O}(h^p)$$

for every $f \in F$.

Local error of Euler's method

One can prove that in the class of continuously differentiable functions Euler's method is first order consistent, that is to say

$$|g_i| \leq \frac{h}{2} \max_{t \in [\xi, \bar{t}]} |x''| = \mathcal{O}(h).$$

Definition

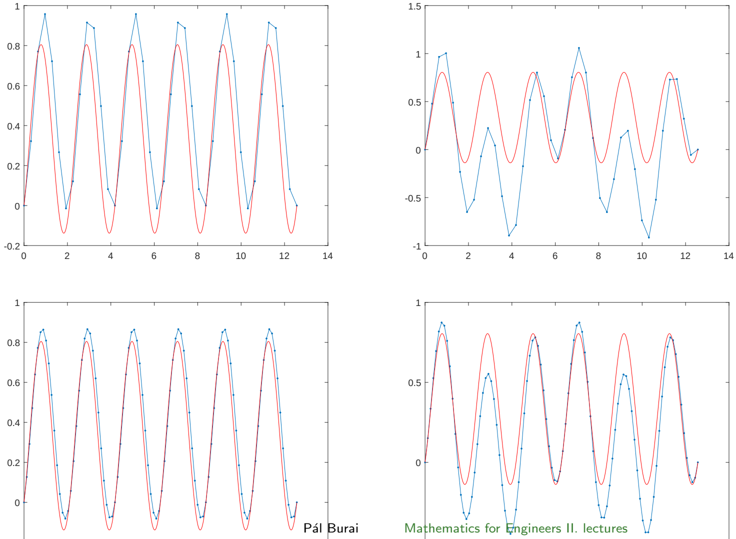
The difference between the exact and the numerical quantity

$$G_i := x(t_i) - x_i$$

is said to be the **global error of Euler's method**.

Taylor series method, Euler's method

The exact and the numerical solutions of the initial value problem $x' = sin(3t) + cos(3t),\ x(0) = 0$ with Euler's method and Taylor series method $(n = 2)$ with 40 and 100 nodes.



Taylor series method, Euler's method

Definition

A numerical method for the initial value problem (1)-(2) is called p**th order convergent in the class** F ($p > 0$), if for all $f \in F$ the global error fulfils

$$|G_i| = \mathcal{O}(h^p).$$

Global error of Euler's method

One can prove that the global error of Euler's method in the class of L Lipschitz functions is

$$|G_i| \leq e^{Lt_i} \left(|G_0| + \sum_{k=0}^{i-1} |g_k| h \right).$$

Runge-Kutta methods

Because of their slow convergence the previous methods applied only for big systems or highly nonlinear systems in practice. One can improve the performance of Euler's method with the following simple modification:

Explicit Runge-Kutta scheme I

We would like to approximate the solution of an initial value problem on an interval $[\xi, \bar{t}\,]$. Let $h := \frac{\bar{t}-\xi}{n}$ for a given $n \in \mathbb{N}$. Then the iteration scheme is the following:

$$\begin{aligned} t_0 &= \xi, & x_0 &= \eta, & t_{i+1} &= t_i + h, \\ k_1 &= f(t_i, x_i), & k_2 &= f\left(t_i + \tfrac{h}{2}, x_i + \tfrac{h}{2} k_1\right), & x_{i+1} &= x_i + h k_2, \end{aligned}$$

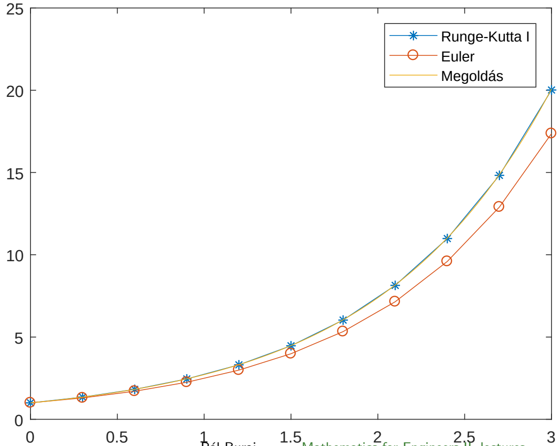
$i = 0, \ldots, n-1.$

Runge-Kutta methods

The approximation of the initial value problem

$$x' = x, \qquad x(0) = 1$$

on the interval $[0, 3]$ with Euler's method and Runge-Kutta scheme I if the partition contains 10 points.



Explicit Runge-Kutta scheme II

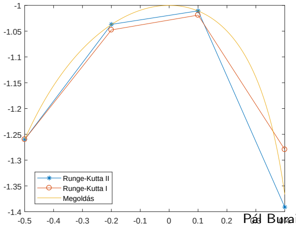
Besides of the earlier assumptions (Runge-Kutta I) we have

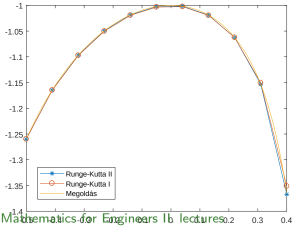
$$t_0 = \xi, \qquad x_0 = \eta, \quad t_{i+1} = t_i + h, \qquad k_1 = f(t_i, x_i),$$

$$k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1\right), \quad k_3 = f\left(t_i + h, x_i - hk_1 + 2hk_2\right),$$

$$x_{i+1} = x_i + \frac{h}{6}(k_1 + 4k_2 + k_3), \qquad\qquad i = 0, \dots, n-1.$$

Numerical solution of initial value problem corresponding to the Bernoulli equation $x' = \frac{-x}{1+t} - (1+t)x^4,\ x(0) = -1$ with 3 and 10 nodes.





Runge-Kutta methods

Explicit Runge-Kutta scheme III

Besides of the earlier assumptions (Runge-Kutta I) we have

$$t_0 = \xi, \quad x_0 = \eta, \quad t_{i+1} = t_i + h, \quad k_1 = f(t_i, x_i), \quad k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2\right), \qquad k_4 = f(t_i + h, x_i + hk_3),$$

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad i = 0, \dots, n-1.$$

General explicit Runge-Kutta scheme

$$k_j = f\left(t + ha_j, x + h\sum_{l=1}^{j-1} b_{j,l}k_l\right), \qquad j = 1, \dots, s,$$

where the constants characterize the method and they are independent from f, from x and from h. One can prove that the magnitude of both the local and the global error is $\mathcal{O}(h^s)$ (assuming reasonable conditions).

Solution of higher order equations

We rewrite an nth order equation into a system of first order ODEs containing n equations. Then we get the separable equation

$$Y' = f(t, Y)$$

(here $Y^T = (y_1, \ldots, y_n)$). One can apply now formally the earlier numerical methods for this equation. **Example:** Let's consider the second order equation $x'' - 8x' + 16x = 0,\ x(0) = 1,\ x'(0) = 5$, then the solution is $x = e^{4t}(1 + t)$. Let's introduce the notations $y_1 = x,\ y_2 = x'$. These functions fulfil the following system of ODEs with the initial condition below.

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= 8y_2 - 16y_1, \quad y_2(0) = 5 \end{aligned}$$

In vector notation

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = Y' = f(t, Y) = \begin{bmatrix} y_2 \\ 8y_2 - 16y_1 \end{bmatrix}, \qquad Y(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Solution of higher order equations

Solution with successive approximation

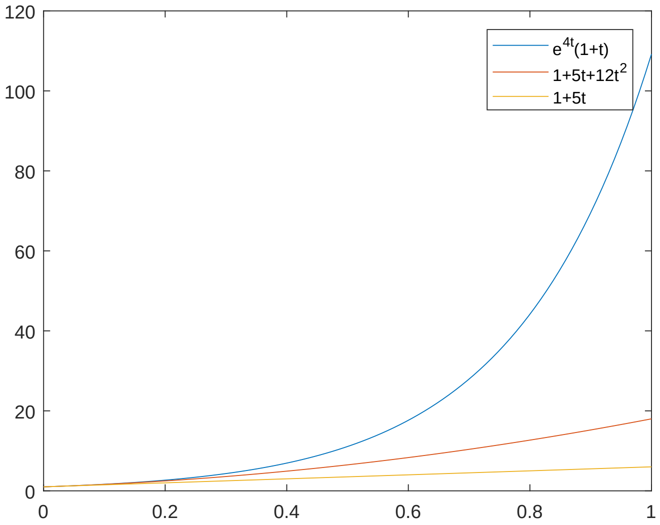
$$Y_0 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad Y_1 = Y_0 + \int_0^t f(s, Y_0) ds = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \int_0^t \begin{bmatrix} 5 \\ 40 - 16 \end{bmatrix} = \begin{bmatrix} 1 + 5t \\ 5 + 24t \end{bmatrix}$$

$$Y_2 = Y_0 + \int_0^t f(s, Y_1) ds = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \int_0^t \begin{bmatrix} 5 + 24s \\ 40 + 192s - 16 - 80s \end{bmatrix} ds$$

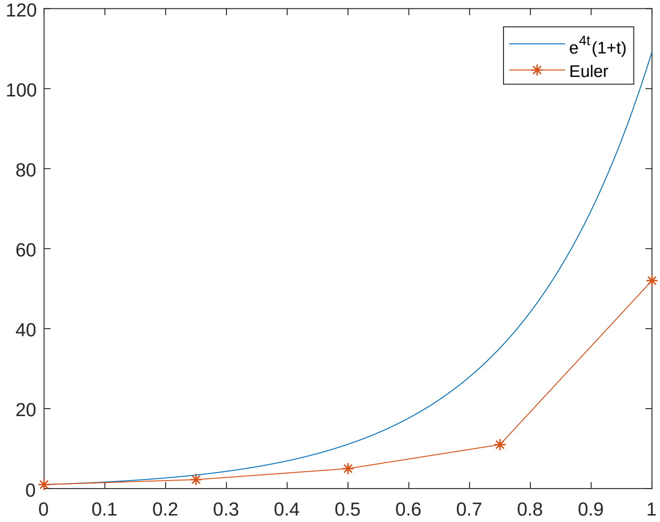
$$Y_2 = \begin{bmatrix} 1 + 5t + 12t^2 \\ 5 + 24t + 56t^2 \end{bmatrix}$$

$$x = y_1 \approx 1 + 5t + 12t^2$$

Solution of higher order equations



Solution with Euler's method on the interval $[0, 1]$ with step size $h = 0.25$:



Project

The projects should be done by groups containing at most three students.

- Numerical solution of an initial value problem involving a nonlinear ODE with two different methods, using different number of nodes on the given interval. Write Matlab code for the solution. Making figures about the approximations. Documentation (5-10 pages).
- Numerical solution of an initial value problem involving a higher order ODE using different number of nodes on the given interval. Write Matlab code for the solution. Making figures about the approximations. Documentation (5-10 pages).